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PRESSURE OF A PLANE CIRCULAR STAMP ON AN ELASTIC HALF-SPACE WITH AN INDENTATION OR INCLUSION

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An axisymmetric mixed problem of the theory of elasticity for a half-space with a hemispherical indentation of radius $\rho < 1$ is considered. The boundary of the half-space is acted upon by a plane circular stamp of unit radius, coaxial with the indentation and covering it completely. There is no friction between the stamp and the half-space. The problem is solved for three cases: the indentation may be empty, or filled with either a perfectly rigid, or a perfectly elastic medium. The solution is constructed in the form of series in terms of the homogeneous solutions of the mixed problem for a half-space, and the

arbitrary constants appearing in the series are obtained from the normal systems of algebraic equations. The concentration of stresses at the stamp edge is studied and a formula obtained relating to the depth of impression of the stamp and the force applied.

1. A plane circular stamp A is pressed without friction by a force T onto an elastic half-space B containing a hemispherical indentation and to the hemisphere C, which may be: (a) empty, (b) perfectly rigid or (c) elastic. The case (b) will be regarded as that of an action exerted by the stamp AC with the frictional forces absent from its whole surface (see Fig. 1).

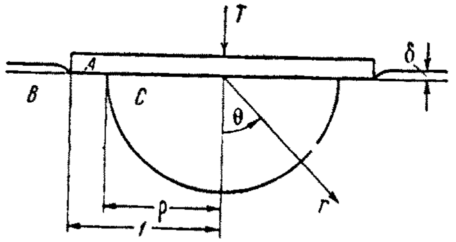

Fig. 1.

Let us construct a subsystem of homogeneous solutions with a singularity at the point $r = 0$, satisfying the following mixed conditions at the boundary $\theta = {}^1/_2\pi$ of the half-space:

$$\tau_{r\theta} = u_\theta = 0 \quad (0 \leqslant r \leqslant 1),$$

$$\tau_{r\theta} = \sigma_\theta = 0 \quad (1 < r < \infty) \tag{1.1}$$

When $\tau_{r\theta} = u_\theta = 0$ on the whole of the boundary plane, the homogeneous solutions for the half-space are given by the zeros of the Legendre function $P_\nu^1(0)$ and have the form [2]

$$2Gu_r^{k1}(r, \theta) = -r^{-2k}\left\{\frac{k(2k+3-4\sigma)}{k-2+2\sigma} P_{2k}(x) A_k + (2k-1) P_{2k-2}(x) B_k\right\}$$
$$2Gu_\theta^{k1}(r, \theta) = r^{-2k}\{P_{2k}^1(x) A_k + P_{2k-2}^1(x) B_k\} \quad (x = \cos\theta) \tag{1.2}$$

Here $k = 1, 2, \ldots$, i.e. the only displacements given here are those which become infinite when $r = 0$. Each element of the given subsystem can be sought as a sum of the solution (1.2), and of the solution of the following mixed problem:

$$\tau_{r\theta} = 0 \quad (0 \leqslant r < \infty,\ \theta = {}^1/_2\pi) \tag{1.3}$$

$$u_\theta = 0 \quad (\theta = {}^1/_2\pi,\ 0 \leqslant r \leqslant 1); \qquad \sigma_\theta = -\sigma_\theta^{k1}(r, {}^1/_2\pi) \quad (\theta = {}^1/_2\pi,\ 1 < r < \infty)$$

The latter solution is obtained in the same manner as the solution of the problem given in [3] and is

$$2Gu_r^{k2}(r, \theta) = \frac{1}{2\pi i}\int_L E_k(\nu)\,[t_2 P_\nu(x) + (\nu+2)(\nu+5-4\sigma) P_{\nu+2}(x)]\, r^{-\nu-2} d\nu$$

$$2Gu_\theta^{k2}(r, \theta) = -\frac{1}{2\pi i}\int_L E_k(\nu)\,[(\nu+1)^{-1} t_2 P_\nu^1(x) + t P_{\nu+2}^1(x)]\, r^{-\nu-2} d\nu \tag{1.4}$$

$$E_k(\nu) = \frac{[\sigma_k^+(\nu) + \sigma_k^-(\nu)]\,\Gamma({}^1/_2 - {}^1/_2\nu)\,\Gamma(1 + {}^1/_2\nu)}{\sqrt{\pi}\,(\nu+1)(2\nu+3)}$$

$$t = \nu - 2 + 4\sigma, \quad t_2 = (\nu+2)^2 - 2(1-\sigma)$$

Here the contour L passes to the left of the straight line $\operatorname{Re}\nu = -2$, while the easily obtained function $\sigma_k^-(\nu)$ is

$$\sigma_k^-(\nu) = -\int_1^\infty \sigma_\theta^{k1}\left(r, \frac{\pi}{2}\right) r^{\nu+2}\, dr =$$

$$= \int_1^\infty \left\{\frac{k[(2k-1)^2-2(1-\sigma)]}{k-2+2\sigma} P_{2k}(0)A_k + (2k-1)^2 P_{2k-2}(0) B_k\right\} r^{\nu-2k+1}\,dr = \tag{1.5}$$

$$= \frac{(-1)^{k+1}(2k-1)!!}{(\nu+2-2k)(2k-2)!!}\left\{\frac{(2k-1)^2-2(1-\sigma)}{2(k-2+2\sigma)} A_k - (2k-1)B_k\right\}$$

The unknown functions

$$\sigma_k^+(\nu) = \int_0^1 \sigma_\theta^{k2}\left(r, \frac{\pi}{2}\right) r^{\nu+2}\,dr, \quad u_k^-(\nu) = \int_1^\infty u_\theta^{k2}\left(r, \frac{\pi}{2}\right) r^{\nu+1}\,dr \tag{1.6}$$

satisfy the Wiener-Hopf equation

$$\sigma_k^+(\nu) + \sigma_k^-(\nu) = K(\nu)\,u_k^-(\nu) \tag{1.7}$$

$$K(\nu) = -\frac{G(1+\nu)(2+\nu)\Gamma(-1/2\nu)\Gamma(3/2+1/2\nu)}{2(1-\sigma)\Gamma(2+1/2\nu)\Gamma(1/2-1/2\nu)}$$

Using the factorization [3]

$$K(\nu) = \frac{K^-(\nu)}{K^+(\nu)}, \quad K^-(\nu) = -\frac{G(1+\nu)(2+\nu)\Gamma(-1/2\nu)}{2(1-\sigma)\Gamma(1/2-1/2\nu)}$$

$$K^+(\nu) = \frac{\Gamma(2+1/2\nu)}{\Gamma(3/2+1/2\nu)} \tag{1.8}$$

we transform (1.7) and introduce the function [4]

$$J_k(\nu) = [\sigma_k^+(\nu) + \sigma_k^-(\nu)]\,K^+(\nu) - \sigma_k^-(\nu)\,K^+(2k-2) =$$

$$= u_k^-(\nu)\,K^-(\nu) - \sigma_k^-(\nu)\,K^+(2k-2) \tag{1.9}$$

regular over the whole ν -plane.

Estimating the growth of the terms which takes place when $|\nu| \to \infty$ with the aid of the Stirling formula for the gamma function and of the generalized Liouville theorem we find that $J_k(\nu) = C_k$. We therefore have, in accordance with (1.9), (1.5) and (1.8)

$$\sigma_k^+(\nu) + \sigma_k^-(\nu) = \tag{1.10}$$

$$= \frac{\Gamma(3/2+1/2\nu)}{\Gamma(2+1/2\nu)}\left\{C_k - \frac{(-1)^k 2k}{\sqrt{\pi}(\nu+2-2k)}\left[\frac{(2k-1)^2-2(1-\sigma)}{2(k-2+2\sigma)} A_k - (2k-1)B_k\right]\right\}$$

For the fruitful utilization of the homogeneous solutions of (1.1) it is important that they are self-balancing when $r > 1$. This condition can be fulfilled as the constant C_k can be chosen in an arbitrary manner. Expanding the integrals in (1.4) into series in residues with $r > 1$ we find that the principal stress vectors different from zero will contain stresses generated by only two poles, namely $\nu = 2k - 2$ and $\nu = -1$. The first ones are balanced by the solution (1.2) and the residues at the point $\nu = -1$ vanish when $\sigma_k^+(-1) + \sigma_k^-(-1) = 0$, i.e. by (1.10) together with

$$C_k = \frac{(-1)^k 2k}{\sqrt{\pi}(1-2k)}\left[\frac{(2k-1)^2-2(1-\sigma)}{2(k-2+2\sigma)} A_k - (2k-1)B_k\right] \tag{1.11}$$

Combining the solutions (1.2) and (1.4) and computing the stresses we finally obtain $(k = 1, 2, \ldots)$

$$2Gu_r^{(k)}(r, \theta) = -r^{-2k}\left[\frac{k(2k+3-4\sigma)}{k+2\sigma-2} P_{2k}(x)A_k + (2k-1)P_{2k-2}(x)B_k\right] +$$

$$+ \frac{1}{2\pi i}\int_L E_k(\nu)\,[t_2 P_\nu(x) + (\nu+2)(\nu+5-4\sigma)P_{\nu+2}(x)]\,r^{-\nu-2}\,d\nu$$

$$2Gu_0^{(k)}(r,\theta) = r^{-2k}\,[P_{2k}^1(x)\,A_k + P_{2k-2}^1(x)\,B_k] - \tag{1.12}$$
$$- \frac{1}{2\pi i}\int_L E_k(\nu)\,[t_2(\nu+1)^{-1}P_\nu^1(x) + tP_{\nu+2}^1(x)]\,r^{-\nu-2}\,d\nu$$

$$\sigma_\theta^{(k)}(r,\theta) = -r^{-2k-1}\Big\{\frac{(k-1)\,[(2k-1)^2 - 2(1-\sigma)]}{k-2+2\sigma}P_{2k}(x)\,A_k +$$
$$+ (2k-1)^2 P_{2k-2}(x)\,B_k + \operatorname{ctg}\theta\,[P_{2k}^1(x)\,A_k + P_{2k-2}^1(x)\,B_k]\Big\} +$$
$$+ \frac{1}{2\pi i}\int_L E_k(\nu)\,\{t_2[(\nu+1)P_\nu(x) + (\nu+1)^{-1}\operatorname{ctg}\theta P_\nu^1(x)] -$$
$$- (\nu+2)[(\nu+1)^2 - 2(1-\sigma)]\,P_{\nu+2}(x) - t\operatorname{ctg}\theta P_{\nu+2}^1(x)\}\,r^{-\nu-3}\,d\nu$$

$$\tau_{r\theta}^{(k)}(r,\theta) = -r^{-2k-1}\Big[\frac{2k^2-1+\sigma}{k-2+2\sigma}P_{2k}^1(x)\,A_k + 2kP_{2k-2}^1(x)\,B_k\Big] +$$
$$+ \frac{1}{2\pi i}\int_L E_k(\nu)\,t_2\Big[\frac{\nu+2}{\nu+1}P_\nu^1(x) + P_{\nu+2}^1(x)\Big]\,r^{-\nu-3}\,d\nu$$

$$\sigma_r^{(k)}(r,\theta) = 2r^{-2k-1}k\Big[\frac{(2k+1)(k+1)-1-\sigma}{k-2+2\sigma}P_{2k}(x)\,A_k + (2k-1)\,P_{2k-2}(x)\,B_k\Big] -$$
$$- \frac{1}{2\pi i}\int_L E_k(\nu)(\nu+2)\,\{t_2P_\nu(x) + [(\nu+2)(\nu+5) - 2\sigma]\,P_{\nu+2}(x)\}\,r^{-\nu-3}\,d\nu$$

$$\sigma_\varphi^{(k)}(r,\theta) + \sigma_\theta^{(k)}(r,\theta) =$$
$$= -2r^{-2k-1}k\Big[\frac{(k-1)(2k+1)+2-4k\sigma}{k-2+2\sigma}P_{2k}(x)\,A_k + (2k-1)\,P_{2k-2}(x)\,B_k\Big] +$$
$$+ \frac{1}{2\pi i}\int_L E_k(\nu)(\nu+2)\,\{t_2P_\nu(x) + [\nu(\nu+3) + 4 - 4\sigma(\nu+2)]\,P_{\nu+2}(x)\}\,r^{-\nu-3}\,d\nu$$

$$E_k(\nu) = \frac{(-1)^k k(\nu+1)\,\{[(2k-1)^2 - 2(1-\sigma)]\,A_k - 2(2k-1)(k-2+2\sigma)\,B_k\}}{(2k-1)(k-2+2\sigma)(2k-2-\nu)(\nu+2)(2\nu+3)\cos(^1/_2\pi\nu)} \tag{1.13}$$

Comparison of these expressions with (1.41) of [3] shows that they represent the homogeneous solution of (1.1) with the principal vector $T \neq 0$ if we set $k = 0$, $A_0 = B_0 = 0$ and

$$E_0(\nu) = -\frac{T}{4(2+\nu)(2\nu+3)\cos(^1/_2\pi\nu)} \tag{1.14}$$

The normal stresses and displacements at the boundary near the line separating the conditions can be obtained, as in [1], with the help of contour integration and asymptotic estimates and have the following form ($k \geqslant 1$):

$$\sigma_0^{(k)}\Big(r,\frac{\pi}{2}\Big) \sim -\frac{4(-1)^k k}{\pi(2k-1)\sqrt{2(1-r)}}\times$$
$$\times\Big\{\frac{(2k-1)^2 - 2(1-\sigma)}{2(k-2+2\sigma)}A_k - (2k-1)\,B_k\Big\} \quad \text{for } r\to 1-0$$

$$u_\theta^{(k)}\Big(r,\frac{\pi}{2}\Big) \sim \frac{4(1-\sigma)k(-1)^k\sqrt{2(r-1)}}{\pi G(2k-1)}\times$$
$$\times\Big\{\frac{(2k-1)^2 - 2(1-\sigma)}{2(k-2+2\sigma)}A_k - (2k-1)\,B_k\Big\} \quad \text{for } r\to 1+0$$

$$\sigma_\theta^{(0)}\Big(r,\frac{\pi}{2}\Big) \sim -\frac{T}{2\pi\sqrt{2(1-r)}},\qquad u_0^{(0)}\Big(r,\frac{\pi}{2}\Big) \sim \frac{T(1-\sigma)\sqrt{2(r-1)}}{2\pi G} \tag{1.15}$$

2. We seek the solutions of the problems (a) and (b) determined by the initial conditions

$$\tau_{r0} = u_\theta = 0 \quad (\rho \leqslant r \leqslant 1,\ \theta = {}^1/_2\pi) \quad \tau_{r\theta} = \sigma_\theta = 0 \quad (1 < r < \infty,\ \theta = {}^1/_2\pi) \quad (2.1)$$

$$\text{a)}\ \tau_{r\theta} = \sigma_r = 0, \quad \text{б)}\ \tau_{r\theta} = u_r = 0 \quad (0 \leqslant \theta \leqslant {}^1/_2\pi,\ r = \rho) \quad (2.2)$$

in the form of series

$$u_\theta = \sum_{k=0}^{\infty} u_\theta^{(k)}(r, \theta), \qquad u_r = \sum_{k=0}^{\infty} u_r^{(k)}(r, \theta) \quad (2.3)$$

which at once satisfy the conditions (2.1) and the condition of equilibrium. The Jordan lemma and the theorem of residues make possible the replacement of the integrals in (1.12), in the case when $r = \rho$ with the residue series written in terms of the negative zeros of the function $\cos({}^1/_2\pi\nu)$. This yields

$$2Gu_\theta^k(\rho, \theta) = \rho^{-2k}\left[P_{2k}^1(x) A_k + P_{2k-2}^1(x) B_k\right] + \quad (2.4)$$

$$+ \sum_{n=0}^{\infty} E_k^*(n)\left\{\frac{(2n+1)^2 - 2(1-\sigma)}{2n+2} P_{2n+2}^1(x) + (2n+5-4\sigma) P_{2n}^1(x)\right\}\rho^{2n+1}$$

$$2Gu_r^{(k)}(\rho, \theta) = -\rho^{-2k}\left[\frac{k(2k+3-4\sigma)}{k-2+2\sigma} P_{2k}(x) A_k + (2k-1) P_{2k-2}(x) B_k\right] +$$

$$+ \sum_{n=0}^{\infty} E_k^*(n)\{[(2n+1)^2 - 2(1-\sigma)] P_{2n+2}(x) + (2n+1)(2n-2+4\sigma) P_{2n}(x)\}\rho^{2n+1} \quad (2.5)$$

$$\tau_{r0}^{(k)}(\rho, \theta) = -\rho^{-2k-1}\left[\frac{2k^2-1+\sigma}{k-2+2\sigma} P_{2k}^1(x) A_k + 2kP_{2k-2}^1(x) B_k\right] + \quad (2.6)$$

$$+ \sum_{n=0}^{\infty} E_k^*(n)\left[(2n+1)^2 - 2(1-\sigma)\right]\left[\frac{2n+1}{2n+2} P_{2n+2}^1(x) + P_{2n}^1(x)\right]\rho^{2n}$$

$$\sigma_r^{(k)}(\rho, \theta) = 2\rho^{-2k-1} k\left[\frac{2k^2+3k-\sigma}{k-2+2\sigma} P_{2k}(x) A_k + (2k-1) P_{2k-2}(x) B_k\right] + \quad (2.7)$$

$$+ \sum_{n=0}^{\infty} E_k^*(n)(2n+1)\left[(4n^2+4n-1+2\sigma) P_{2n+2}(x) + \right.$$

$$\left. + (4n^2-2n-2-2\sigma) P_{2n}(x)\right]\rho^{2n}$$

$$E_k^*(n) = $$

$$= \frac{4(-1)^{k+n+1} k(n+1)\left[(4k^2-4k-1+2\sigma) A_k - 2(2k-1)(k-2+2\sigma) B_k\right]}{\pi(2k-1)(2n+1)(2k+2n+1)(k-2+2\sigma)(4n+3)} \quad (k \geqslant 1)$$

$$E_0^*(n) = \frac{(-1)^n T}{2\pi(2n+1)(4n+3)} \quad (2.8)$$

Let us insert the formulas (2.5) - (2.7) into solutions (2.3) and conditions (2.2) and change the order of summation in the double series. Comparing the coefficients of the functions $P_{2k}(x)$ and $P_{2k}^1(x)$ $(k = 0, 1, \ldots)$ we obtain the following infinite systems of algebraic equations:

$$a_0 X_{1,4} + \sum_{n=1}^{\infty} f_{0,n}\rho^{2n+1} X_{n,3} = g_0 \quad (2.9)$$

$$b_k^{(s)}(X_{k,3} + X_{k,4}) + a_k^{(s)} X_{k+1,4} + \sum_{n=1}^{\infty} f_{k,n}^{(s)}\rho^{2k+2n+1} X_{n,3} = g_k^{(s)} \quad (2.10)$$

$$(s = 1, 2;\ k = 1, 2, \ldots)$$

Here for both problems we have

$$X_{k,3} = \rho^{-2k-1}\left[\frac{(2k-1)^2 - 2(1-\sigma)}{(2k-1)(k-2+2\sigma)} A_k - 2B_k\right], \quad X_{k,4} = 2\rho^{-2k-1} B_k \qquad (2.11)$$

$$a_0 = {}^1/_2, \qquad a_k^{(1)} = k+1, \qquad b_k^{(1)} = \frac{(2k-1)(2k^2-1+\sigma)}{(2k-1)^2 - 2(1-\sigma)}$$

$$g_k^{(1)} = \frac{T(-1)^k \rho^{2k}}{2\pi}\left[\frac{(2k+1)^2 - 2(1-\sigma)}{(2k+1)(4k+3)} - \frac{(2k-1)^2 - 2(1-\sigma)}{\rho^2 2k(4k-1)}\right]$$

$$f_{k,n}^{(1)} = \frac{(-1)^{k+n+1} 2n}{\pi}\left\{\frac{(2k+2)[(2k+1)^2 - 2(1-\sigma)]}{(2k+1)(4k+3)(2k+2n+1)} - \frac{(2k-1)^2 - 2(1-\sigma)}{\rho^2(4k-1)(2k+2n-1)}\right\}$$

Moreover, for the problem (a) we have

$$a_k^{(2)} = -(k+1)(2k+1)$$

$$g_k^{(2)} = \frac{T(-1)^k \rho^{2k}}{2\pi}\left[\frac{2k(2k-1) - 2(1+\sigma)}{4k+3} - \frac{(2k-1)^2 - 2(1-\sigma)}{\rho^2(4k-1)}\right] \qquad (2.12)$$

$$g_0 = \frac{T(1+\sigma)}{3\pi}, \qquad b_k^{(2)} = -\frac{2k(2k-1)[(2k+1)(k+1) - (1+\sigma)]}{(2k-1)^2 - 2(1-\sigma)}$$

$$f_{0,n} = \frac{4(-1)^{n+1}(1+\sigma)n}{3\pi(2n+1)}$$

$$f_{k,n}^{(2)} = \frac{(-1)^{k+n+1} 2n}{\pi}\left\{\frac{(2k+2)[(2k-1)2k - 2(1+\sigma)]}{(4k+3)(2k+2n+1)} - \frac{2k[(2k-1)^2 - 2(1-\sigma)]}{\rho^2(4k-1)(2k+2n-1)}\right\}$$

and for the problem (b)

$$a_k^{(2)} = \frac{2k+1}{2}, \qquad g_k^{(2)} = \frac{T(-1)^k \rho^{2k}}{2\pi}\left[\frac{2k-2+4\sigma}{4k+3} - \frac{(2k-1)^2 - 2(1-\sigma)}{\rho^2(4k-1)(2k-1)}\right] \qquad (2.13)$$

$$g_0 = -\frac{T(1-2\sigma)}{3\pi}, \qquad b_k^{(2)} = \frac{k(2k-1)(2k+3-4\sigma)}{(2k-1)^2 - 2(1-\sigma)}, \qquad f_{0,n} = \frac{8(-1)^n(1-2\sigma)n}{3\pi(2n+1)}$$

$$f_{k,n}^{(2)} = \frac{2n(-1)^{k+n+1}}{\pi}\left\{\frac{(2k+2)(2k-2+4\sigma)}{(4k+3)(2k+2n+1)} - \frac{2k[(2k-1)^2 - 2(1-\sigma)]}{\rho^2(4k-1)(2k-1)(2k+2n-1)}\right\}$$

Let us reduce the system (2.9) and (2.10) to its canonical form. Eliminating from (2.10) the unknowns $X_{k,4}$ and $X_{k+1,4}$ with $s = 1, 2$ respectively, we obtain ($k = 1, 2, \ldots$)

$$(a_k^{(1)} b_k^{(2)} - a_k^{(2)} b_k^{(1)}) X_{k+1,4} + \sum_{n=1}^{\infty} (f_{k,n}^{(1)} b_k^{(2)} - f_{k,n}^{(2)} b_k^{(1)}) \rho^{2k+2n+1} X_{n,3} =$$
$$= g_k^{(1)} b_k^{(2)} - g_k^{(2)} b_k^{(1)} \qquad (2.14)$$

$$(a_k^{(2)} b_k^{(1)} - a_k^{(1)} b_k^{(2)})(X_{k,3} + X_{k,4}) + \sum_{n=1}^{\infty} (f_{k,n}^{(1)} a_k^{(2)} - f_{k,n}^{(2)} a_k^{(1)}) \rho^{2k+2n+1} X_{n,3} =$$
$$= g_k^{(1)} a_k^{(2)} - g_k^{(2)} a_k^{(1)} \qquad (2.15)$$

Next we use (2.15) to eliminate the unknown $X_{1,4}$ from (2.9). We replace k in (2.15) with $k+1$ and eliminate $X_{k+1,4}$ from (2.14). This yields ($k = 1, 2, \ldots$)

$$a_0 (b_1^{(1)} a_1^{(2)} - b_1^{(2)} a_1^{(1)}) X_{1,3} +$$
$$+ \sum_{n=1}^{\infty} [a_0 (f_{1,n}^{(1)} a_1^{(2)} - f_{1,n}^{(2)} a_1^{(1)}) \rho^2 + f_{0,n} (b_1^{(2)} a_1^{(1)} - a_1^{(2)} b_1^{(1)})] \rho^{2n+1} X_{n,3} =$$
$$= a_0 (a_1^{(2)} g_1^{(1)} - a_1^{(1)} g_1^{(2)}) + g_0 (b_1^{(2)} a_1^{(1)} - b_1^{(1)} a_1^{(2)}) \qquad (2.16)$$

$$(a_{k+1}^{(2)} b_{k+1}^{(1)} - a_{k+1}^{(1)} b_{k+1}^{(2)})(a_k^{(1)} b_k^{(2)} - a_k^{(2)} b_k^{(1)}) X_{k+1,3} +$$

$$+\sum_{n=1}^{\infty}[(f_{k+1,n}^{(1)}a_{k+1}^{(2)}-f_{k+1,n}^{(2)}a_{k+1}^{(1)})(a_k^{(1)}b_k^{(2)}-b_k^{(1)}a_k^{(2)})-$$
$$-\rho^{-2}(f_{k,n}^{(1)}b_k^{(2)}-f_{k,n}^{(2)}b_k^{(1)})(b_{k+1}^{(1)}a_{k+1}^{(2)}-b_{k+1}^{(2)}a_{k+1}^{(1)})]\,\rho^{2k+2n+3}X_{n,3}=$$
$$=(g_{k+1}^{(1)}a_{k+1}^{(2)}-g_{k+1}^{(2)}a_{k+1}^{(1)})(a_k^{(1)}b_k^{(2)}-a_k^{(2)}b_k^{(1)})+(g_k^{(1)}b_k^{(2)}-g_k^{(2)}b_k^{(1)})(b_{k+1}^{(2)}a_{k+1}^{(1)}-b_{k+1}^{(1)}a_{k+1}^{(2)})$$

Obviously the double series of the matrix of this system converges absolutely for all $\rho < 1$ and the moduli of its free terms are bounded. Thus the system (2.16) is related to the normal Poincaré-Koch systems [5]. By a verbatim repetition of the arguments from the Sect. 2 of [1] we conclude that in both cases, (a) and (b), solutions of this system exist, are unique and can be obtained using the Cramer method. When k is large we have A_k and $B_k \sim k\rho^{4k}$.

It must be noted that the system (2.16) only contains the unknowns $X_{k,3}$; the unknowns $X_{k,4}$ are obtained from (2.9) and (2.14). This makes it possible to obtain a solution for the truncated system (2.15) with twice the number of the correct signs as compared with the solution obtained from the system (2.9), (2.10) truncated at the terms of the same order. We also note that the problem (a) can easily be generalized to the case in which arbitrary loads are applied to the spherical part of the boundary of the helf-space, while the problem (b) can be generalized to the case in which the spherical part of the stamp AC differs somewhat from the indentation, i.e. in which $u_r = f(\theta)$ when $r = \rho$. In the latter case the function $f(\theta)$, which appears in the condition (2.2) must be expanded into a series in Legendre polynomials of even degree, otherwise the procedure remains the same.

In conclusion let us write a formula connecting the depth of the indentation δ made by the stamp with the force T applied

$$2G\delta = 2Gu_r\big|_{\substack{\theta=0\\ r=\infty}} = \tag{2.17}$$
$$=\lim_{r\to\infty}\sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_L E_k(\nu)\,[t_2P_\nu(1)+(\nu+2)(\nu+5-4\sigma)\,P_{\nu+2}(1)]\,r^{-\nu-3}\,d\nu=$$
$$=-\frac{T(1-\sigma)}{2}-\sum_{k=1}^{\infty}\frac{(1-\sigma)(-1)^k\{[(2k-1)^2-2(1-\sigma)]\,A_k-2(2k-1)(k-2+2\sigma)B_k\}}{(2k-1)(k-2+2\sigma)}$$

3. Let us consider the case (c). Retaining the previous G and σ for the region B, we denote the constants for the region C by G_1 and σ_1. In the region B we shall seek the solutions in the form of (2.3), and in C in the form of series in homogeneous solutions satisfying the conditions $\tau_{r\theta} = u_\theta = 0$ when $\theta = {}^1/_2\pi$ with a singularity at infinity [2]

$$2G_1u_r=\sum_{k=0}^{\infty}\left[\frac{(2k+1)(2k-2+4\sigma_1)\,r^{2k+1}}{2k+5-4\sigma_1}C_k+2kr^{2k-1}D_k\right]P_{2k}(x)$$
$$2G_1u_\theta=\sum_{k=1}^{\infty}[r^{2k+1}C_k+r^{2k-1}D_k]\,P_{2k}^1(x) \tag{3.1}$$

In this case both the condition of equilibrium and the boundary conditions

$$u_\theta=\tau_{r\theta}=0\quad(0\leqslant r\leqslant 1,\ \theta={}^1/_2\pi),\qquad \sigma_\theta=\tau_{r\theta}=0\quad(1<r<\infty,\ \theta={}^1/_2\pi)$$

of the problem (c) will be satisfied. Let us now comply with the four conditions of conjugacy of the solutions (2.3) and (3.1) on the hemisphere $0 \leqslant \theta \leqslant {}^1/_2\pi,\ r = \rho\ (\rho < 1)$.

Using the expansions (2.4) - (2.7) and comparing again the factors accompanying the functions $P_{2k}(x)$ and $P_{2k}{}^1(x)$, we obtain the following system of equations:

$$\sum_{p=1}^{4} d_{kp}^{(i)} X_{k,p} + d_k^{(i)} X_{k+1,4} + \sum_{n=1}^{\infty} \varphi_{k,n}^{(i)} \rho^{2k+2n+1} X_{n,3} = h_k^{(i)} \qquad (k = 0, 1, \ldots) \tag{3.2}$$

When $k = 0$ $i = 1, 2$, while for $k \geqslant 1$ we have $i = 1, 2, 3, 4$ and the following notation introduced

$$X_{k,1} = \rho^{2k} C_k, \quad X_{k,2} = \rho^{2k-2} D_k, \quad X_{k,3} = \rho^{-2k-1}\left[\frac{(2k-1)^2 - 2(1-\sigma)}{(2k-1)(k-2+2\sigma)} A_k - 2B_k\right]$$

$$X_{k,4} = 2\rho^{-2k-1} B_k, \quad d_{01}^{(1)} = -\frac{1-2\sigma_1}{G_1(5-4\sigma_1)}, \quad d_{02}^{(1)} = d_{03}^{(1)} = d_{04}^{(1)} = 0$$

$$d_{02}^{(2)} = d_{03}^{(2)} = d_{04}^{(2)} = 0$$

$$d_0^{(1)} = \frac{1}{4G}, \quad \varphi_{0,n}^{(1)} = \frac{4(-1)^n (1-2\sigma) n}{3\pi G(2n+1)}, \quad h_0^{(1)} = -\frac{T(1-2\sigma)}{6\pi G} \tag{3.3}$$

$$d_{01}^{(2)} = \frac{1+\sigma_1}{5-4\sigma_1}, \quad d_0^{(2)} = {}^1/_2, \quad \varphi_{0n}^{(2)} = \frac{4(-1)^{n+1}(1+\sigma) n}{3\pi(2n+1)}, \quad h_0^{(2)} = \frac{T(1+\sigma)}{3\pi}$$

$$d_{k1}^{(1)} = d_{k2}^{(1)} = -\frac{1}{2G_1}, \quad d_{k3}^{(1)} = d_{k4}^{(1)} = \frac{(2k-1)(k-2+2\sigma)}{2G[(2k-1)^2 - 2(1-\sigma)]}, \quad d_k^{(1)} = \frac{1}{4G}$$

$$\varphi_{kn}^{(1)} = \frac{(-1)^{k+n} n}{\pi G}\left[\frac{(2k+2)(2k+5-4\sigma)}{(4k+3)(2k+3)(2k+2n+1)} + \right.$$

$$\left. + \frac{(2k-1)^2 - 2(1-\sigma)}{\rho^2(4k-1)(2k-1)(2k+2n-1)}\right]$$

$$h_k^{(1)} = \frac{T(-1)^{k+1}\rho^{2k}}{4\pi G}\left[\frac{2k+5-4\sigma}{(4k+3)(2k+1)} - \frac{(2k-1)^2 - 2(1-\sigma)}{\rho^2(4k-1)(2k-1)2k}\right]$$

$$d_{k1}^{(2)} = \frac{(2k+1)(2k-2+4\sigma_1)}{(2k+5-4\sigma_1)2G_1}, \qquad d_{k2}^{(2)} = \frac{k}{G_1}$$

$$d_{k3}^{(2)} = d_{k4}^{(2)} = \frac{k(2k-1)(2k+3-4\sigma)}{2G[(2k-1)^2 - 2(1-\sigma)]}, \qquad d_k^{(2)} = \frac{2k+1}{4G}$$

$$\varphi_{k,n}^{(2)} = \frac{(-1)^{k+n+1} n}{\pi G}\left[\frac{(2k+2)(2k-2+4\sigma)}{(4k+3)(2k+2n+1)} - \frac{2k[(2k-1)^2 - 2(1-\sigma)]}{\rho^2(4k-1)(2k-1)(2k+2n-1)}\right]$$

$$h_k^{(2)} = \frac{T(-1)^k \rho^{2k}}{4\pi G}\left[\frac{2k-2+4\sigma}{4k+3} - \frac{(2k-1)^2 - 2(1-\sigma)}{\rho^2(4k-1)(2k-1)}\right]$$

$$d_{k1}^{(3)} = \frac{(2k+1)[2k(2k-1) - 2(1+\sigma_1)]}{2k+5-4\sigma_1}, \qquad d_{k2}^{(3)} = 2k(2k-1)$$

$$d_k^{(3)} = -\frac{(2k+2)(2k+1)}{2}, \quad d_{k3}^{(3)} = d_{k4}^{(3)} = -\frac{2k(2k-1)[(2k+1)(k+1) - (1+\sigma)]}{(2k-1)^2 - 2(1-\sigma)}$$

$$\varphi_{kn}^{(3)} = \frac{(-1)^{k+n+1} 2n}{\pi}\left\{\frac{4(k+1)[k(2k-1) - 1 - \sigma]}{(4k+3)(2k+2n+1)} - \frac{2k[(2k-1)^2 - 2(1-\sigma)]}{\rho^2(4k-1)(2k+2n-1)}\right\}$$

$$h_k^{(3)} = \frac{T(-1)^k \rho^{2k}}{2\pi}\left[\frac{2k(2k-1) - 2(1+\sigma)}{4k+3} - \frac{(2k-1)^2 - 2(1-\sigma)}{\rho^2(4k-1)}\right]$$

$$d_k^{(4)} = k+1, \qquad d_{k1}^{(4)} = \frac{(2k+1)^2 - 2(1-\sigma_1)}{2k+5-4\sigma_1}$$

$$d_{k2}^{(4)} = (2k-1), \qquad d_{k3}^{(4)} = d_{k4}^{(4)} = \frac{(2k-1)(2k^2-1+\sigma)}{(2k-1)^2-2(1-\sigma)}$$

$$\varphi_{kn}^{(4)} = \frac{(-1)^{k+n+1}2n}{\pi}\left[\frac{(2k+2)[(2k+1)^2-2(1-\sigma)]}{(2k+1)(4k+3)(2k+2n+1)} - \frac{(2k-1)^2-2(1-\sigma)}{\rho^2(4k-1)(2k+2n-1)}\right]$$

$$h_k^{(4)} = \frac{T(-1)^k\rho^{2k}}{2\pi}\left[\frac{(2k+1)^2-2(1-\sigma)}{(2k+1)(4k+3)} - \frac{(2k-1)^2-2(1-\sigma)}{\rho^2 2k(4k-1)}\right]$$

Eliminating now the unknowns $X_{k,1}$ and $X_{k,2}$, from the corresponding groups of Eqs. (3.2), each group corresponding to a single value of k, we obtain (2.9) and (2.10) in which

$$a_0 = d_{01}^{(2)}d_0^{(1)} - d_{01}^{(1)}d_0^{(2)}, \qquad f_{0n} = \varphi_{0,n}^{(1)}d_{01}^{(2)} - \varphi_{0,n}^{(2)}d_{01}^{(1)}, \qquad g_1^{(1)} = h_0^{(1)}d_{01}^{(2)} - h_0^{(2)}d_{01}^{(1)}$$

$$a_k^{(s)} = \sum_{i=1}^4 d_k^{(i)}D_{ks}^{(i)}, \qquad b_k^{(s)} = \sum_{i=1}^4 d_{k3}^{(i)}D_{ks}^{(i)}, \qquad f_{k,n}^{(s)} = \sum_{i=1}^4 \varphi_{kn}^{(i)}D_{ks}^{(i)}, \qquad g_k^{(s)} = \sum_{i=1}^4 h_k^{(i)}D_{ks}^{(i)}$$

Here $D_{ks}^{(i)}$ denote the algebraic complements of the elements $d_{ks}^{(i)}$

$$\begin{vmatrix} d_{k2}^{(1)} & d_{k2}^{(2)} & 0 & 0 \\ d_{k1}^{(1)} & d_{k1}^{(2)} & 0 & 0 \\ 0 & 0 & -d_{k2}^{(3)} & -d_{k2}^{(4)} \\ 0 & 0 & -d_{k1}^{(3)} & -d_{k1}^{(4)} \end{vmatrix}$$

Thus in the case (c) only one fourth of the unknowns, namely the ones entering the solutions (2.3) and (3.1), need be obtained from the normal system of algebraic equations (2.16). The depth of impression of the stamp is again given by (2.17).

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